Math 250

## 5.2 Definite Integrals

Briggs

Objectives

1) Understand the difference between a definite integral

$$\int_a^b f(x)\,dx \quad = \quad \text{area of plane region, (a number)}$$

and an indefinite integral

$$\int f(x)\,dx \quad = \quad \text{antiderivative of } f(x) \text{ (a function)}$$

2) Understand the definition of a definite integral.

3) Evaluate a definite integral using geometry formulas for area of plane region

4) Evaluate a definite integral using properties of definite integrals.

* Do NOT need to evaluate a definite integral using limits of summation formulas

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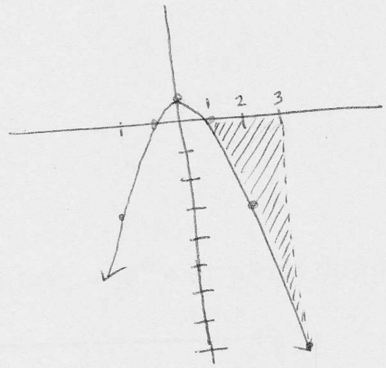
① Evaluate the left Riemann sum for $f(x) = 1 - x^2$ on $[1, 3]$ with $n = 4$ equally-spaced subintervals.

Notice the graph:



$\Delta x = \frac{3-1}{4} = \frac{1}{2}$

Find partition

$1 \quad \frac{3}{2} \quad 2 \quad \frac{5}{2} \quad 3$

$$\text{area} \approx \Delta x \left[ f(x_0) + f(x_1) + f(x_2) + f(x_3) \right]$$

$f(x_4)$ not used for left endpoints.

$$= \frac{1}{2}\left[ f(1) + f\left(\tfrac{3}{2}\right) + f(2) + f\left(\tfrac{5}{2}\right) \right]$$

$$= \boxed{\frac{-19}{4}} = \boxed{-4.75}$$

Negative area?!

True area is always positive.

But using function values $f(x_k^*)$ below the x-axis makes the Riemann sum negative (or a portion of the sum uses neg values that add to other values that are positive).

So the Riemann sum does not approximate true area, but <u>Net Area</u> (sometimes called <u>Net Signed Area</u>)

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When did approximation of the area of the plane region in 5.1, we were careful to specify

1) the location of $x_k^*$ in the subinterval
   - left Riemann sum
   - right Riemann sum
   - midpoint Riemann sum

2) the width of each subinterval was constant $\Delta x$ to make a regular or uniform partition.

But a Riemann sum does not require either of these limitations, so long as

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b,$$

$\Delta x_k = x_k - x_{k-1}$ width of the $k^{th}$ subinterval can be different from other widths

$x_k^*$ can be *any* point in the subinterval $[x_{k-1}, x_k]$.

$$\sum_{k=1}^{n} \Delta x_k \cdot f(x_k^*)$$ is a Riemann sum. (general)

Because our goal is the exact net signed area of the plane region, we need a way to take a limit, even if the $\Delta x_k$ are all different sizes.

Definition: $\Delta$ = maximum of all $\Delta x_k$.

So if $\Delta \to 0$ then all $\Delta x_k \to 0$

And we take a limit over all possible partitions:

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} \Delta x_k \cdot f(x_k^*)$$ = exact net signed area of plane region, *if* the limit exists.

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<u>Defn</u>: A function $f$ is <u>integrable</u> on $[a,b]$ if

$\lim_{\Delta \to 0} \sum_{k=1}^{n} \Delta x_k \cdot f(x_k^*)$ exists and gives the same (unique) value no matter the partition.

<u>Defn</u>: The <u>definite integral</u> of $f$ over $[a,b]$ is written

$$\int_a^b f(x)\,dx$$

and is defined to be the limit

$$\int_a^b f(x)\,dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} \Delta x_k \cdot f(x_k^*).$$

and its value is the net signed area of the plane region bounded by the $x$-axis, the graph $y=f(x)$ and the vertical lines $x=a$ and $x=b$.

<u>Theorem</u>:
If $f$ is continuous on $[a,b]$
then $f$ is integrable on $[a,b]$.

<u>Theorem</u>:
If $f$ is bounded on $[a,b]$ ← {Not infinite, Not unbounded, no vertical asymptotes}
and $f$ has a finite # of discontinuities on $[a,b]$

Then $f$ is integrable on $[a,b]$.

A function is <u>bounded</u> on an interval if there is a positive number $M$ so that

$$-M \le f(x) \le M$$

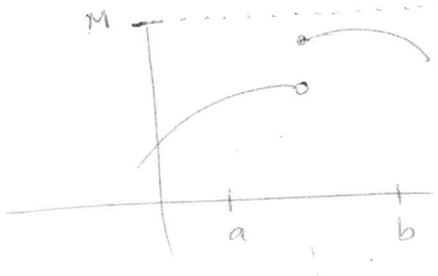
for all $x$ on the interval.

(Effectively, this means the function is finite.)

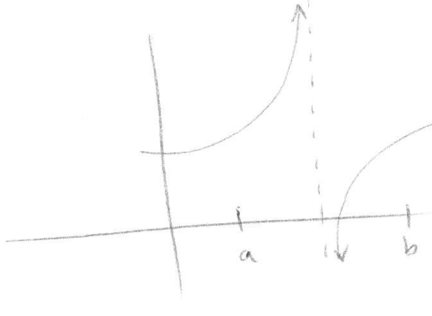
<u>Integrable Functions</u>

$f$ is defined on $[a,b]$ where $a$ and $b$ are finite.

a) If $f$ has finitely many discontinuities in $[a,b]$ and is bounded on $[a,b]$, then $f$ is integrable.




This $f$ is integrable




This $f$ is not integrable

b) If $f$ is not bounded on $[a,b]$, $f$ is not integrable.

# Notation for the definite integral

$$\int_a^b f(x)\,dx$$

dx is the <u>differential</u>, telling us
- x is the variable of integration
- the integrand ends here.

a is called the <u>lower limit of integration</u>

f(x) is called the <u>integrand</u>

"the definite integral of f from a to b"

b is called the <u>upper limit of integration</u>

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CAUTION:

$$\int_a^b f(x)\,dx \;=\; \text{net area} \;=\; \text{a number value}$$

$$\int f(x)\,dx \;=\; \text{general antiderivative} \;=\; \text{a family a functions}$$

$F(x)+C$ so
$F'(x)=f(x)$

These are <u>not</u> interchangeable.

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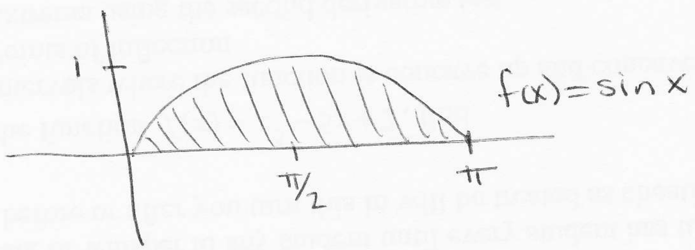
(2) Example: Write this net area as a definite integral:




$$\boxed{\int_0^{\pi} \sin x\,dx}$$

Note: the variable letter chosen is irrelevant -- the area will be the same if we write:

$$\int_0^{\pi} \sin(t)\,dt$$

We call the variable of integration a <u>dummy variable</u>.

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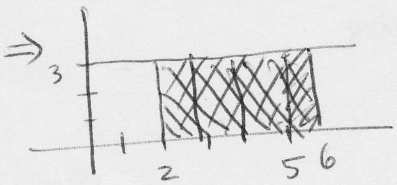
Find areas using formulas from geometry. Sketch graph, shade area, and find area using geometry formula

(3) $\int_2^6 3\,dx$



area of rectangle $= L \cdot W$

$= (6-2)\cdot 3$

$= 4 \cdot 3$

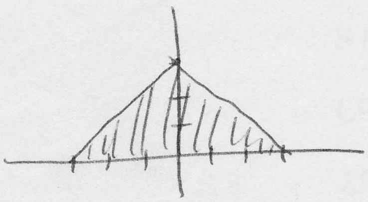
$= \boxed{12}$

(4) $\int_{-3}^{3} \left(3-|x|\right) dx$

area of triangle $= \frac{1}{2}\cdot b \cdot h$

$= \frac{1}{2}(3-(-3))\cdot 3$

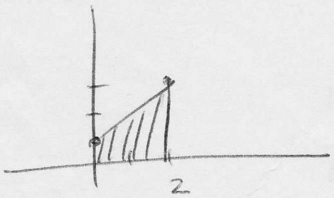
$= \frac{1}{2}(6)\cdot 3$



$= \boxed{9}$

(5) $\int_0^2 (x+1)\,dx$

area of trapezoid $= \frac{1}{2}(B_1+B_2)h$



$= \frac{1}{2}(1+3)\cdot 2$

$= \frac{1}{2}(4)(2)$

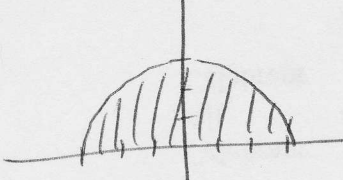
$= \boxed{4}$

$\int_0^2 x-2\,dx$

(6) $\int_{-3}^{3} \sqrt{9-x^2}\,dx$

area of semicircle $= \frac{1}{2}\pi r^2$

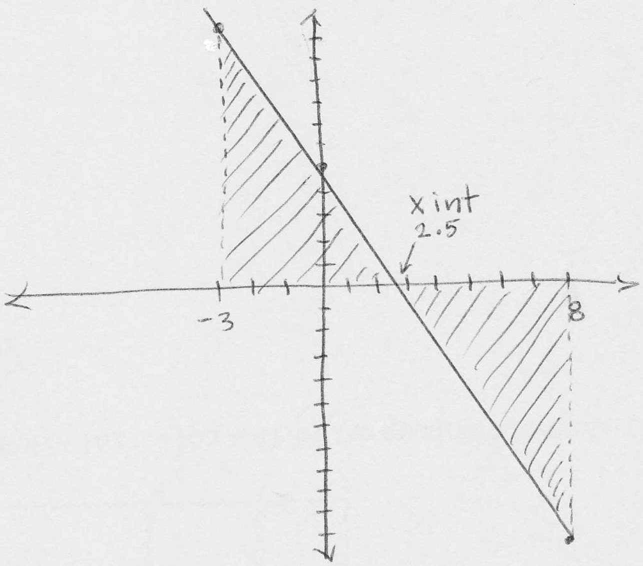
$= \frac{1}{2}\pi (3)^2$



$= \boxed{\frac{9}{2}\pi}$

Note: Even after this, geometry formulas will continue to be our preferred (or even only) method for evaluating some definite integrals.

(7) Find the area and the net area of region bounded by $f(x) = -2x+5$ on $[-3, 8]$ using geometry.




positive area $[-3, 2.5]$

area of triangle $= \frac{1}{2} B \cdot H$

$= \frac{1}{2}(2.5-(-3)) \cdot f(-3)$

$= \frac{1}{2}(5.5)(11)$

$= 30.25$

negative area $[2.5, 8]$

area of triangle $= \frac{1}{2} B \cdot H$

$= \frac{1}{2}(8-2.5) \cdot f(8)$

$= \frac{1}{2}(5.5) \cdot (-11)$

$= -30.25$

$$\boxed{\begin{array}{l} \text{true area} = 30.25 + 30.25 = 60.5 \\ \text{net area} = 30.25 - 30.25 = 0 \end{array}}$$

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## Properties of Definite Integrals:

1) $\int_a^a f(x)\,dx = 0$ area of width zero

2) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ backward $\Rightarrow$ negative

3) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

if c is in between a and b — divide into two areas.

if c is not, divide into one positive and one negative area

4) $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$ scalar multiple

5) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$. addition + Subtraction.

6) if f is integrable and nonnegative on $[a,b]$
then $0 \le \int_a^b f(x)\,dx$.

7) If f and g are integrable on $[a,b]$
and $f(x) \le g(x)$ $\forall x \in [a,b]$
then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

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Given

$$\int_0^3 f(x)\,dX = 5$$

$$\int_3^6 f(x)\,dx = -2$$

$$\int_3^6 g(X) = 4$$

Evaluate the definite integrals

(8) $$\int_0^6 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^6 f(x)\,dx$$

$$= 5 + (-2)$$

$$= \boxed{3}$$

(9) $$\int_6^0 f(x) = -\int_6^0 f(x) = \boxed{-3}$$

(10) $$\int_3^6 \big(f(x) + g(x)\big)\,dx = \int_3^6 f(x)\,dx + \int_3^6 g(x)\,dx$$

$$= -2 + 4$$

$$= \boxed{2}$$

(11) $$\int_6^3 \big(g(x) - f(x)\big)\,dx = -\int_3^6 [g(x) - f(x)]\,dx$$

$$= -\int_3^6 g(x)\,dx + \int_3^6 f(x)\,dx$$

$$= -4 + -2$$

$$= \boxed{-6}$$

(12) $$\int_3^6 [-2f(x) + 7g(x)]\,dx = -2\int_3^6 f(x)\,dx + 7\int_3^6 g(x)\,dx$$

$$= -2(-2) + 7(4)$$

$$= 4 + 28$$

$$= \boxed{32}$$

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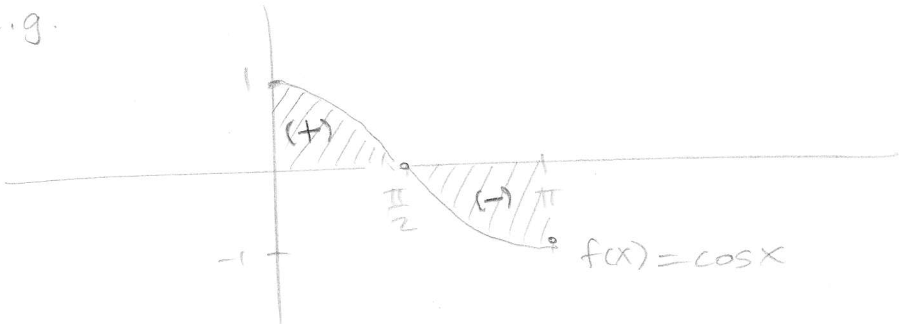
<u>Theorem</u>: If $f$ is continuous on $[a,b]$
{but not always non-negative}

then

$$\int_a^b f(x)\,dx = \text{net signed area between } f(x) \text{ and x-axis } [a,b].$$

e.g.




$$\int_0^{\pi} \cos x\,dx = 0$$

because area above the x-axis is positive
and area below the x-axis is negative!